Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

Spring 2025 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

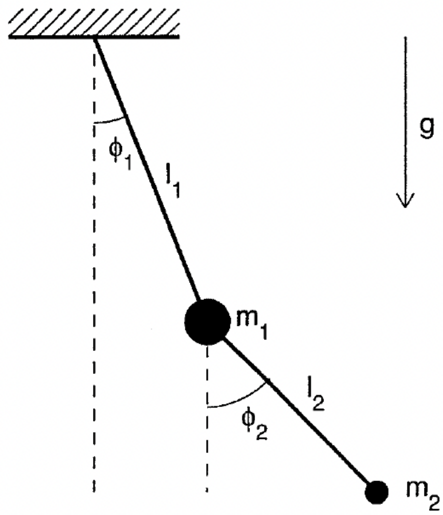
Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

Write your ID number (not your name!) on each exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

CLASSICAL MECHANICS 1

Double pendulum



A double pendulum consists of one pendulum attached to another. Consider a double pendulum with masses m_1 and m_2 attached by rigid massless wires of length $l_1 = l_2 = l$. Let $\phi_{1,2}$ denote the angles that each wire makes with the vertical, as shown in the figure. Let g denote the acceleration due to gravity. The motion is in the plane shown and frictionless.

(a)[6pt] Derive the Lagrangian. Express your result in terms of l, m_1, m_2, g and ϕ_1, ϕ_2 and their derivatives.

(b)[2pt] Determine the generalized momenta p_1, p_2 conjugate to ϕ_1, ϕ_2.

(c)[4pt] Determine the Euler-Lagrange equations of motion of ϕ_1 and ϕ_2.

(d)[6pt] Assume small oscillations, $|\phi_1|, |\phi_2| \ll 1$. Determine the characteristic frequencies.

(e)[2pt] What are the limiting values of the frequencies in the case $m_1 \gg m_2$? Interpret your result physically.

Solution

(a) The kinetic energy is given by:

$$\text{K.E.} = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) \tag{1}$$

From the figure,

$$x_1 = l\sin\phi_1,\ y_1 = -l\cos\phi_1,\ x_2 = l(\sin\phi_1 + \sin\phi_2),\ y_2 = -l(\cos\phi_1 + \cos\phi_2), \tag{2}$$

which implies

$$\dot{x}_1 = l\cos\phi_1\dot{\phi}_1,\ \dot{y}_1 = l\sin\phi_1\dot{\phi}_1,\ x_2 = l\left(\cos\phi_1\dot{\phi}_1 + \cos\phi_2\dot{\phi}_2\right),\ y_2 = l\left(\sin\phi_1\dot{\phi}_1 + \sin\phi_2\dot{\phi}_2\right) \tag{3}$$

Thus, in terms of the angular variables, the kinetic energy becomes:

$$\text{K.E.} = \frac{1}{2}\left(m_1 + m_2\right)l^2\dot{\phi}_1^2 + \frac{1}{2}m_2 l^2\dot{\phi}_2^2 + m_2 l^2\cos(\phi_1 - \phi_2)\dot{\phi}_1\dot{\phi}_2 \tag{4}$$

The potential energy in terms of the angular variables is:

$$\text{P.E.} = -gl\cos\phi_1(m_1 + m_2) - glm_2\cos\phi_2 \tag{5}$$

Thus the Lagrangian is given by:

$$\boxed{\mathcal{L} = \frac{1}{2}\left(m_1 + m_2\right)l^2\dot{\phi}_1^2 + \frac{1}{2}m_2 l^2\dot{\phi}_2^2 + m_2 l^2\cos(\phi_1 - \phi_2)\dot{\phi}_1\dot{\phi}_2 + gl\cos\phi_1(m_1 + m_2) + glm_2\cos\phi_2} \tag{6}$$

(b) The generalized momenta are given by:

$$\boxed{p_1 = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = \left(m_1 + m_2\right)l^2\dot{\phi}_1 + m_2 l^2\dot{\phi}_2\cos(\phi_1 - \phi_2)} \tag{7}$$

and

$$\boxed{p_2 = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_2} = m_2 l^2\dot{\phi}_2 + m_2 l^2\dot{\phi}_1\cos(\phi_1 - \phi_2)} \tag{8}$$

(c) The Euler-Lagrange equations are given by:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_i}\right) = \frac{\partial\mathcal{L}}{\phi_i} \tag{9}$$

Applied to ϕ_1 this yields:

$$\frac{d}{dt}\left(\left(m_1 + m_2\right)l^2\dot{\phi}_1 + m_2 l^2\dot{\phi}_2\cos(\phi_1 - \phi_2)\right) = -m_2 l^2\sin(\phi_1 - \phi_2)\dot{\phi}_1\dot{\phi}_2 - gl\sin\phi_1(m_1 + m_2) \tag{10}$$

Taking the derivative:

$$\begin{aligned}(m_1+m_2)l^2\ddot{\phi}_1 + m_2l^2\ddot{\phi}_2\cos(\phi_1-\phi_2) - m_2l^2\dot{\phi}_2(\dot{\phi}_1-\dot{\phi}_2)\sin(\phi_1-\phi_2)\\ = -m_2l^2\sin(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2 - gl\sin\phi_1(m_1+m_2),\end{aligned} \tag{11}$$

which simplifies to:

$$\boxed{(m_1+m_2)\,\ddot{\phi}_1 + m_2\ddot{\phi}_2\cos(\phi_1-\phi_2) + m_2\dot{\phi}_2^2\sin(\phi_1-\phi_2) = -\frac{g}{l}\sin\phi_1(m_1+m_2)} \tag{12}$$

Applying the Euler-Lagrange equation to ϕ_2 yields:

$$\frac{d}{dt}\Big(m_2l^2\dot{\phi}_2 + m_2l^2\dot{\phi}_1\cos(\phi_1-\phi_2)\Big) = m_2l^2\sin(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2 - glm_2\sin\phi_2 \tag{13}$$

Taking the derivative:

$$\begin{aligned}m_2l^2\ddot{\phi}_2 + m_2l^2\ddot{\phi}_1\cos(\phi_1-\phi_2) - m_2l^2\dot{\phi}_1\left(\dot{\phi}_1-\dot{\phi}_2\right)\sin(\phi_1-\phi_2)\\ = m_2l^2\sin(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2 - glm_2\sin\phi_2,\end{aligned} \tag{14}$$

which simplifies to

$$\boxed{\ddot{\phi}_2 + \ddot{\phi}_1\cos(\phi_1-\phi_2) - \dot{\phi}_1^2\sin(\phi_1-\phi_2) = -\frac{g}{l}\sin\phi_2} \tag{15}$$

(d) Expanding the Euler-Lagrange equations to order $\ddot{\phi}_i$ yields:

$$\begin{aligned}(m_1+m_2)\ddot{\phi}_1 + m_2\ddot{\phi}_2 + \frac{g}{l}(m_1+m_2)\phi_1 = 0\\ \ddot{\phi}_2 + \ddot{\phi}_1 + \frac{g}{l}\phi_2 = 0\end{aligned} \tag{16}$$

These second order differential equations should have a solution of the form $\phi_i(t) = A_i\cos\omega t$. Plugging into Eq. (16) yields the matrix equation:

$$\begin{pmatrix}\left(\frac{g}{l}-\omega^2\right)(m_1+m_2) & -\omega^2 m_2\\ -\omega^2 & -\omega^2+\frac{g}{l}\end{pmatrix}\begin{pmatrix}A_1\cos\omega t\\ A_2\cos\omega t\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{17}$$

Solutions exist when the determinant of the matrix on the right-hand-side vanishes, i.e.,

$$\left(\frac{g}{l}-\omega^2\right)^2(m_1+m_2) - \omega^4 m_2 = 0 \tag{18}$$

Combining terms:

$$\omega^4 m_1 - 2\frac{g}{l}\omega^2(m_1+m_2) + \left(\frac{g}{l}\right)^2(m_1+m_2) = 0, \tag{19}$$

which has solutions:

$$\boxed{\omega_{\pm}^2 = \frac{1}{m_1}\frac{g}{l}\sqrt{m_1+m_2}\left(\sqrt{m_1+m_2} \pm \sqrt{m_2}\right)} \tag{20}$$

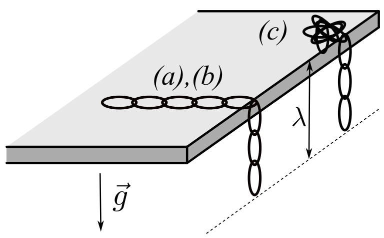
(e) In the case $m_1 \gg m_2$,

$$\boxed{\omega_{\pm}^2 \to \frac{g}{l}} \tag{21}$$

This corresponds to the mass m_1 being fixed and m_2 oscillating about m_1 with the usual frequency of a single pendulum, $\omega_0 = g/l$.

CLASSICAL MECHANICS 2

Falling chain



A chain of length L and uniform mass per unit length ρ is perfectly flexible and cannot stretch. Initially ($t = 0$), it is lying at rest on a horizontal table, with a segment segment of length λ ($< L$) hanging over the edge. In parts (a) and (b), the horizontal part of the chain is aligned straight and away (perpendicular) from the edge of the table, while in part (c) it is piled up very close to the edge.

(a)[6pt] First, assume that there is no friction between the chain and the table. At time $t = 0$, the chain is released and starts to slide off the table: its vertical segment $y(t)$ increases with time $t > 0$ while the horizontal segment slides towards the edge.

i. Using energy conservation, find a relationship between y, $\dot{y}$, L, g (free-fall acceleration) and λ. Does it depend on ρ? why?

ii. Determine (in terms of L, g, and λ) the speed of the chain when $y = L$ (*i.e.*, when the last segment of the chain has just left the horizontal surface).

iii. Obtain an equation of motion $\ddot{y}(t) = f(y)$ and, for $y(t) \leq L$, show that the solution for $y(t)$ takes the form $y(t) = \lambda\,\Phi(t)$. Determine the function Φ.

iv. How does the motion of the chain change when $t > \sqrt{L/g}\,\cosh^{-1}(L/\lambda)$? Sketch your result for $y(t)$ including both $y < L$ and $y \geq L$.

(b)[6pt] Now suppose that there is "viscous" friction force between the chain and the table $F = 2fL\dot{y}$, where f is a constant. Show that a general solution of the corresponding equation of motion takes the form:

$$y(t) = \mathrm{e}^{-st}\left[W_+\,\Theta(t) + W_-\,\Theta(-t)\right],$$

where s, W_+ and W_- are some constants. Find t function Θ and the constant s. Determine the constants $W_\pm$ corresponding to the initial conditions of the problem.

(c)[8pt] Now suppose that the friction is absent but the horizontal segment $(L - \lambda)$ of the chain is compactly piled right on the edge of the table. The other segment $\lambda < L$ hangs over the edge as before, and the chain is released from rest at time $t = 0$.

i. Find the equation of motion for $y(t)$. *Hint: Balance the gravitational force with the rate of change of the chain's momentum.*

ii. Show that in this case, the total mechanical (kinetic + potential) energy of the moving chain) is not conserved, instead decreasing at a rate $C\rho\,\dot{y}^3$ (determine the constant C).

iii. Explain qualitatively the physical origin of the energy dissipation above.

Solution

(a) The chain has length L and uniform mass per unit length ρ.

i. Initially, length $(L-\lambda)$ lies on the table and length λ is hanging. At time t, length $(L-y(t))$ lies on the table and length $y(t)$ is hanging. Measure potential energies from a reference height of the table top. Then, by conservation of energy applied initially (*i.e.*, at time $t=0$) and at time t:

$$\frac{1}{2}\rho\,L\,\dot{y}^2-\rho\,y\,g\,\frac{1}{2}\,y=-\rho\,\lambda\,g\,\frac{1}{2}\,\lambda\,.$$

Simplifying and rearranging, we obtain:

$$L\,\dot{y}^2=g\,(\,y^2-\lambda^2\,)\,.$$

ii. Thus, the speed when $y=L$ is given by:

$$\dot{y}\Big|_{y=L}=\sqrt{g\,L}\,\sqrt{1-(\lambda/L)^2}\,.$$

The physical reason why ρ does feature in this result is the equivalence of inertial and gravitational mass.

iii. Take the time-derivative of the energy conservation law, *i.e.*, $L\,\dot{y}^2=g(y^2-\lambda^2)$ and cancel common factors to obtain the linear second-order ordinary differential equation for $y(t)$: $L\,\ddot{y}=g\,y$.
Strategy #1: Solving this (linear, homogeneous) ordinary differential equation by the hypothesis $y(t)=\exp rt$ (*i.e.*, using the fact that the equation has constant coefficients), and implementing the initial conditions $(y(t),\dot{y}(t))\,\big|_{t=0}=(\lambda,0)$, one finds:

$$y(t)=\lambda\,\Phi(\gamma t),\quad\text{with}\quad\gamma=+\sqrt{g/\lambda}\quad\text{and}\quad\Phi(x)=\cosh x\,.$$

Strategy #2: Separation of variables applied to the energy conservation law $L\,\dot{y}^2=g(y^2-\lambda^2)$:

$$\int_{\lambda}^{y(t)}\frac{d\bar{y}}{\sqrt{\bar{y}^2-\lambda^2}}=\sqrt{\frac{g}{L}}\int_0^t d\bar{t}\,.$$

Apply the change of integration variable $\bar{y}=\lambda\cosh\theta$ to obtain:

$$\int_0^{\cosh^{-1}(y(t)/\lambda)}d\theta=\sqrt{\frac{g}{L}}\,t\,,$$

$$\text{or}\quad y(t)=\lambda\cosh\left(\sqrt{g/L}\,t\right).$$

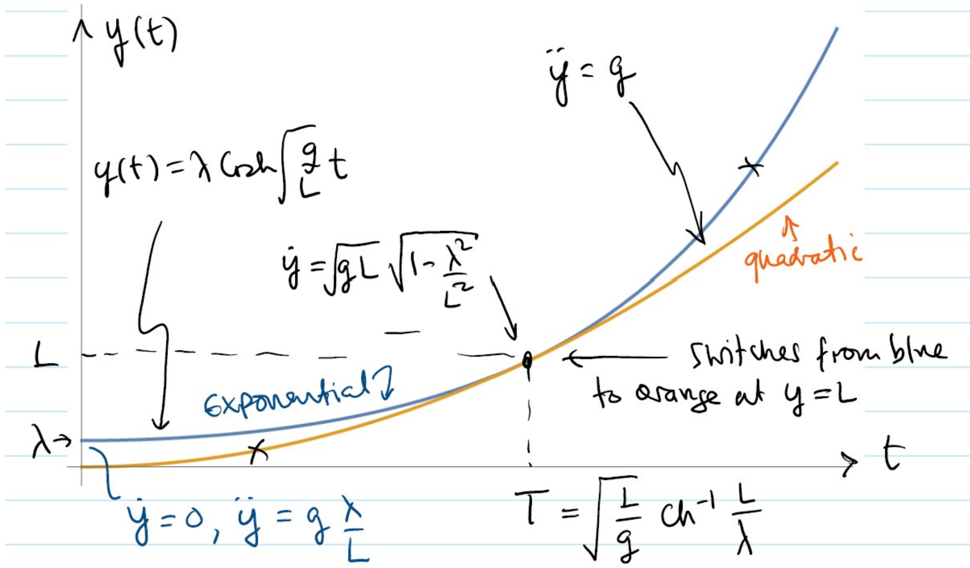


FIG. 1. Part (a.iv) Graph of y versus t.

iv. For $t > \sqrt{L/g}\,\cosh^{-1}(L/\lambda)$ we have $y > L$, so no chain segment remains on the table. Then the whole chain remains vertical and continues to fall, starting from the speed given as the answer to part (ii), but now with constant acceleration g.

(b) With friction the equation of motion becomes: $\rho\, L\, \ddot{y} = \rho\, g\, y - 2 f\, L\, \dot{y}$. Independent solutions of this (homogeneous, linear, constant-coefficient) ordinary differential equation are obtained *via* the substitution $y(t) = \exp rt$. This gives the general solution:

$$y(t) = \mathrm{e}^{-tf/\rho}\left[W_+\,\mathrm{e}^{+t\sqrt{(f/\rho)^2+(g/L)}} + W_-\,\mathrm{e}^{-t\sqrt{(f/\rho)^2+(g/L)}}\right],$$

where W_+ and W_- are constants of integration. Thus, $\Theta(x) = \exp(x)$, $s = (f/\rho)$ and $\Sigma = +\sqrt{(f/\rho)^2 + (g/L)}$.

(c) Now the segment on the table is pooled up into a ball.

i. Balancing the rate of change of momentum with the force gives:

$$\frac{d}{dt}\Big[(\rho\, y)\,\dot{y}\Big] = \Big((\rho\, y)\, g\Big).$$

Thus, on canceling ρ, the equation of motion is found to be:

$$y\,\ddot{y} + \dot{y}^2 = g\,y\,.$$

ii. The rate of change of the energy E is given by:

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}(\rho\, y)\,\dot{y}^2 - (\rho\, y)\, g\,\frac{1}{2}\,y\right].$$

Evaluating the time-derivative and applying the equation of motion gives:

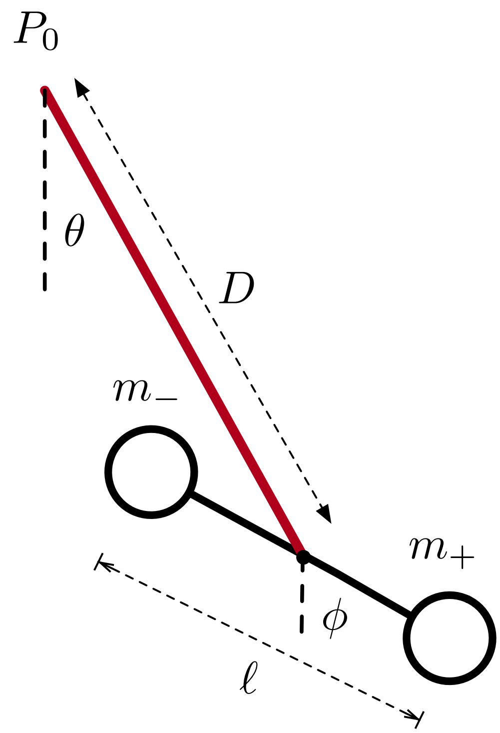
$$\frac{dE}{dt} = -\frac{1}{2}\rho\,\dot{y}^3, \quad \text{so} \quad C = 1/2\,.$$

iii. Minus dE/dt, *i.e.*, $\frac{1}{2}(\rho\,\dot{y})\,\dot{y}^2$, is the power needed to speed up the chain segments from zero (their speed when in the ball) to the speed necessary to take part in the fall of the chain.

CLASSICAL MECHANICS 3

Dumbbell pendulum

Two balls of mass m_+ and m_- are connected by massless rod of length ℓ. The center of the connecting rod is suspended in the earth's gravitational field to a pivot point P_0 by an additional rod of length D as shown below. The two rods can rotate without obstruction, and angles θ and ϕ both will evolve in time. The masses $m_\pm$ are very nearly equal, with $m_\pm = m \pm \delta m$ and $\delta m \ll m$



(a)[6pt] Determine an approximate Lagrangian and action $S[\theta, \phi]$ of the system to leading order in δm.

Hint: Here and below it is useful to consider the center of mass and relative coordinates.

(b)[4pt] Determine the constants of motion for $\delta m = 0$ and $\delta m \neq 0$. In each case identify the symmetry associated with the conservation law.

(c)[4pt] Let $\theta(t)$ and $\phi(t)$ solve the equations of motion. If $\theta(t)$ and $\phi(t)$ are changed as follows:

$$\theta(t) \to \theta(t) + \epsilon(t) \tag{1}$$

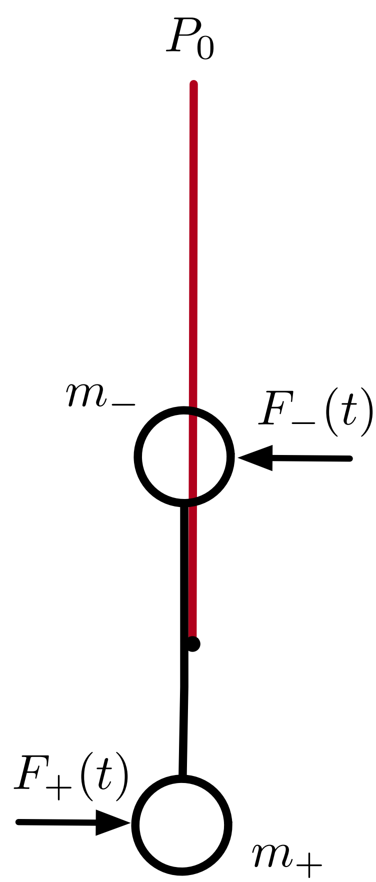
$$\phi(t) \to \phi(t) + \epsilon(t) \tag{2}$$

with $\epsilon(t)$ an arbitrary infinitessimal function, then what is the variation

$$\delta S \equiv S[\theta + \epsilon, \phi + \epsilon] - S[\theta, \phi] \,. \tag{3}$$

What does this specific variation imply for the time evolution of the system? Interpret the result physically and discuss the limit of no gravity.

(d)[6pt] Consider the setup shown below, which is initially in stable equilibrium. At time $t = 0$, a two strongly impulsive forces $F_{\pm}(t) = \pm\kappa_0 \delta(t)$ are applied to the balls (as shown below), causing the connected balls to spin rapidly, i.e. spin with a typical frequency much larger than $\sqrt{g/D}$.



libel=(i) Approximately determine $\phi(t)$ and $\theta(t)$, when $\delta m = 0$.

liibel=(ii) Perturbing the results of the previous item, determine the rapid oscillations of $\theta(t)$ to first order in δm.

Solution

(a) The total mass is independent of δm

$$M = 2m\,. \tag{4}$$

The reduced mass is unchanged to first order

$$\mu = \frac{m_+ m_-}{M} = \frac{(m+\delta m)(m-\delta m)}{M} = \frac{m}{2}\,. \tag{5}$$

A short calculation shows that the center of mass is shifted relative to the center point

$$\Delta x = \frac{1}{M}\left[(m+\delta m)(\ell/2) - (m-\delta m)(\ell/2)\right] = \frac{\delta m \ell}{M} \equiv \alpha\ell\,, \tag{6}$$

where $\alpha \ll 1$.

The (X, Y) coordinates of the center of mass are

$$X = D\sin\theta + \alpha\ell\sin\phi\,, \tag{7}$$
$$Y = -D\cos\theta - \alpha\ell\cos\phi\,. \tag{8}$$

The relative coordinates $\mathbf{r}_+ - \mathbf{r}_-$ are

$$r_x = x_+ - x_- = \ell\sin\phi\,, \tag{9}$$
$$r_y = y_+ - y_- = -\ell\cos\phi\,. \tag{10}$$

The velocities are

$$\dot{X} = D\cos\theta\dot{\theta} + \alpha\ell\cos\phi\dot{\phi}\,, \tag{11}$$
$$\dot{Y} = D\sin\theta\dot{\theta} + \alpha\ell\sin\phi\dot{\phi}\,, \tag{12}$$
$$\dot{r}_x = \ell\cos\phi\dot{\phi}\,, \tag{13}$$
$$\dot{r}_y = \ell\sin\phi\dot{\phi}\,. \tag{14}$$

The kinetic energy of the center of mass is

$$K_{\rm cm} = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2)\,, \tag{15}$$
$$\simeq \frac{1}{2}MD^2\dot{\theta}^2 + MD\alpha\ell\cos(\theta-\phi)\dot{\phi}\dot{\theta}\,. \tag{16}$$

The relative kinetic energy is

$$K_{\rm rel} = \frac{1}{2}\mu\left(\frac{d\mathbf{r}}{dt}\right)^2 = \frac{1}{2}\mu\ell^2\dot{\phi}^2\,. \tag{17}$$

Finally, we have to potential energy

$$U = MgY = -Mg(D\cos\theta + \alpha\ell\cos\phi)\,. \tag{18}$$

So, the Lagrangian has the form

$$L = K_{\rm cm} + K_{\rm rel} - U \tag{19}$$

$$= \frac{1}{2}MD^2\dot\theta^2 + MD\alpha\ell\cos(\theta-\phi)\dot\phi\dot\theta + \frac{1}{2}\mu\ell^2\dot\phi^2 + Mg(D\cos\theta + \alpha\ell\cos\phi)\,. \tag{20}$$

(b) For $\delta m \neq 0$ the only constants is the energy

$$h = K_{\rm cm} + K_{\rm rel} + U\,. \tag{21}$$

following from time translation symmetry. For the $\delta m = 0$, there is time translation symmetry leading to the conservation of energy. In addition the variation

$$\phi \to \phi + \Delta\phi\,, \tag{22}$$

is a symmetry of the action, since this shift does not displace the center of mass. This leads to the conservation of internal angular momentum $L_{\rm int} = \mu\ell^2\dot\phi$.

(c) When there is no gravity, rotations are a symmetry of the action. The specified variation rotates the system as a whole by an angle ϵ around P_0, and, without gravity, leads to angular momentum conservation. Let's see what it gives when there is gravity. Following the Noether method

$$\delta S = \int dt\, \left(\frac{\partial L}{\partial\dot\phi} + \frac{\partial L}{\partial\dot\theta}\right)\partial_t\epsilon + \left(\frac{\partial L}{\partial\phi} + \frac{\partial L}{\partial\theta}\right)\epsilon \tag{23}$$

$$= \int dt\, \left(\frac{\partial L}{\partial\dot\phi} + \frac{\partial L}{\partial\dot\theta}\right)\partial_t\epsilon - mg(D\sin\theta + \alpha\ell\sin\phi)\epsilon\,, \tag{24}$$

leading (after integrating by parts) to the equation of angular momentum evolution

$$\frac{d}{dt}\left(MD^2\dot\theta + MD\alpha\ell\cos(\theta-\phi)(\dot\theta+\dot\phi) + \mu\ell^2\dot\phi\right) = -\,Mg(D\sin\theta + \alpha\ell\sin\phi) \tag{25}$$

$$= -\,MgX_{\rm cm}\,. \tag{26}$$

The right hand side of this equation is the total external torque provided by gravity. If gravity is zero we have a conserved quantity

$$MD^2\dot\theta + MD\alpha\ell\cos(\theta-\phi)(\dot\theta+\dot\phi) + \mu\ell^2\dot\phi = 0\,, \tag{27}$$

which is the total angular momentum of the system around the point P_0.

(d) We first observe that when $\delta m = 0$ we have that the net momentum transfer is zero we have a solution

$$\phi = \omega t + C. \tag{28}$$

The value of ω is related to κ. From the picture, the internal angular momentum

$$\mu\ell^2\dot{\phi} = \kappa_0\ell\,, \tag{29}$$

and so

$$\omega = \frac{\kappa_0}{\mu\ell}\,. \tag{30}$$

Then $\theta = 0$ is the remaining solution.

When this solution is perturbed we have $\theta \ll 1$. Then expanding we look at the conservation equation of angular momentum, which, to first order θ and α, approximately reads

$$\frac{d}{dt}\left(MD^2\dot{\theta} + MD\alpha\ell\cos(\omega t)\omega + \mu\ell^2\omega\right) = -MgD\theta - Mg\alpha\ell\sin(\omega t)\,, \tag{31}$$

leading to

$$\ddot{\theta} + \Omega^2\theta = -\alpha\frac{\ell}{D}\sin(\omega t)\left(\Omega^2 - \omega^2\right)\,. \tag{32}$$

The dominant term is the second one

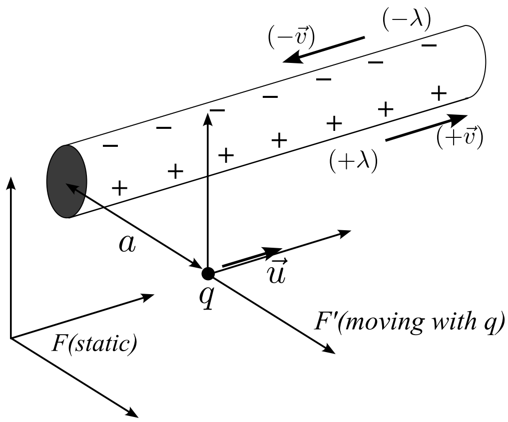
$$\ddot{\theta} + \Omega^2\theta = \alpha\frac{\ell}{D}\sin(\omega t)\omega^2\,. \tag{33}$$

This is a driven harmonic oscillator, the steady state solution is found by substituting $A\sin(\omega t)$ leading to

$$A = -\frac{\alpha\frac{\ell}{D}\omega^2}{\omega^2 - \Omega^2} \simeq -\alpha\frac{\ell}{D}\,. \tag{34}$$

ELECTROMAGNETISM 1

Magnetism from relativity



Consider a thin string of positive charges moving to the right with velocity v. We will characterize the stream of charges by a continuous line charge λ. Superimposed on this positive string, is a negative string with line charge $-\lambda$ moving to the left at the same speed. At a distance a away, there is a point charge q moving to the right at a speed $u < v$ as shown in the Figure.

(a)[2pt] What is the net charge and current carried by the strings of charges, in the rest frame F?

(b)[5pt] What is the net charge λ' carried by the strings of charges, in the frame F' attached to the charge q?

(c)[5pt] Evaluate the electric force exerted on the charge q in the frame F'.

(d)[5pt] What is the force exerted on the charge q in the frame F?

(e)[3pt] Identify explicitly the physical origin of the force in F.

Solution

(a) The net line charge is zero $\lambda - \lambda = 0$, and the net current in the frame F is

$$I = \lambda(+v) + (-\lambda)(-v) = 2\lambda v \qquad \text{to the right} \tag{1}$$

(b) In the frame F' attached to the charge q, the net line charge is not zero $\lambda_\pm = \pm\gamma_\pm\lambda$ with

$$\gamma_\pm = \frac{1}{\sqrt{1 - V_\pm^2/c^2}} = \gamma_v\gamma_u\left(1 \mp \frac{uv}{c^2}\right) \qquad V_\pm = \frac{v \mp u}{1 \mp uv/c^2} \tag{2}$$

hence the net line charge

$$\lambda' = \lambda_+ + \lambda_- = (\gamma_+ - \gamma_-)\lambda_0 = -\frac{2\lambda_0 uv/c^2}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}} \equiv \gamma_v\gamma_u(-2\lambda_0 uv/c^2) \tag{3}$$

Here λ_0 is the line charge of the positive string in its own rest frame, i.e $\lambda = \gamma_v\lambda_0$.

(c) In the frame F', the net λ' creates an electric force on the charge q

$$\mathcal{F}' = qE' = q\frac{2\lambda'}{a} = \frac{2q}{a}\left(\frac{-2\lambda uv/c^2}{\sqrt{1 - u^2/c^2}}\right) \tag{4}$$

(d) The force $\mathcal{F}'$ translates by relativity to an attractive force $\mathcal{F}$ in the frame F

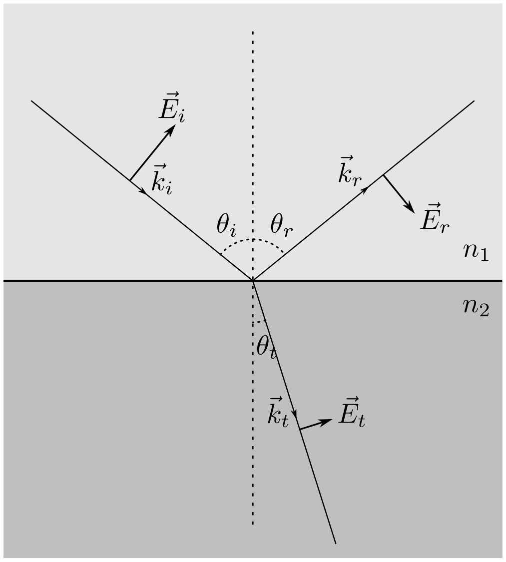
$$\mathcal{F} = \frac{\mathcal{F}'}{\gamma_v} = \frac{-4q\lambda uv}{ac^2} \tag{5}$$

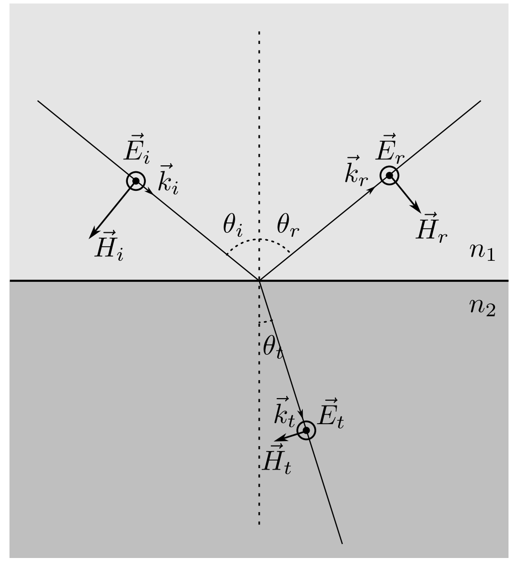
(e) The attraction of the charge q to the wire is the expected Lorentz force

$$\mathcal{F} = -\frac{qv}{c}\left(\frac{4\lambda u}{ac} = B\right) \tag{6}$$

ELECTROMAGNETISM 2

Reflection and Polarization





(a)[5pt] Consider a plane electromagnetic wave incident on a boundary between two dielectric materials. What components of the fields (**D**, **B**, **E**, and **H**) are continuous across the boundary? Hint: Consider the integral form of Maxwell's equations at the boundary.

(b)[5pt] Using these relationships, find the ratio of the electric field amplitudes of electromagnetic plane waves incident and reflected from a planar boundary between two materials with indices of refraction given by n_1 and n_2 as a function of angle of incidence, θ_i. Consider (p-)polarization where the electric field is in the plane of incidence/reflection (and the magnetic field is parallel to the boundary) as shown in the diagram on the left. You may find it helpful to use Snell's law ($n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$), and you can assume that the two media have the same magnetic properties, $\mu_1 = \mu_2$. Also, you can make use of the relationship $k = n\omega/c$ between the wave-number k, the frequency ω, and the refraction index n, as well as the relationship between the field strengths in the plane waves, $H = kE/\mu\omega$.

(c)[5pt] Find the reflection coefficient for the other (s-)polarization in which the magnetic field is in the plane of incidence/reflection (and the electric field is parallel to the boundary) as shown in the figure on the right.

(d)[5pt] For what angle and polarization does the reflection coefficient vanish if $n_1 = 1.2$ and $n_2 = 1.5$?

Solution

(a) From Gauss' law, we have that the perpendicular component of $\mathbf{D}$ is continuous. From Faraday's law, we have that the parallel component of $\mathbf{E}$ is continuous. Similarly, the parallel component of $\mathbf{H}$ is continuous, and the perpendicular component of $\mathbf{B}$ is continuous.

(b) Making use of $\theta_r = \theta_i$ and the field continuities given above, we have for p polarization:

$$H_i - H_r = H_t \tag{1}$$

$$E_i \cos(\theta_i) + E_r \cos(\theta_i) = E_t \cos(\theta_t) \tag{2}$$

Making use of $H = kE/\mu\omega$, we can combine the two equations and eliminate E_t to arrive at:

$$r_p = \frac{E_r}{E_i} = \frac{k_i \cos(\theta_t) - k_t \cos(\theta_i)}{k_t \cos(\theta_i) + k_i \cos(\theta_t)} \tag{3}$$

Using Snell's law to express θ_t in terms of θ_i and the relationship between n and k, we can rewrite this equation as:

$$r_p = \frac{E_r}{E_i} = \frac{\sqrt{(n_2/n_1)^2 - \sin^2(\theta_i)} - (n_2/n_1)^2 \cos(\theta_i)}{\sqrt{(n_2/n_1)^2 - \sin^2(\theta_i)} + (n_2/n_1)^2 \cos(\theta_i)} \tag{4}$$

(c) Making use of $\theta_r = \theta_i$ and the field continuities given above, we have for s polarization:

$$E_i + E_r = E_t \tag{5}$$

$$H_i \cos(\theta_i) - H_r \cos(\theta_i) = H_t \cos(\theta_t) \tag{6}$$

Making use of $H = kE/\mu\omega$, we can combine the two equations and eliminate E_t to arrive at:

$$r_s = \frac{E_r}{E_i} = \frac{k_i \cos(\theta_i) - k_t \cos(\theta_t)}{k_i \cos(\theta_i) + k_t \cos(\theta_t)} \tag{7}$$

Using Snell's law to express θ_t in terms of θ_i and the relationship between n and k, we can rewrite this equation as:

$$r_s = \frac{E_r}{E_i} = \frac{\cos(\theta_i) - \sqrt{(n_2/n_1)^2 - \sin^2(\theta_i)}}{\cos(\theta_i) + \sqrt{(n_2/n_1)^2 - \sin^2(\theta_i)}} \tag{8}$$

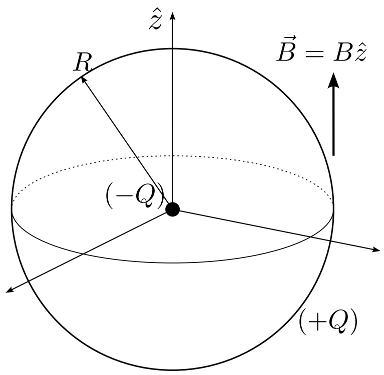
(d) The reflection coefficient vanishes for p polarization at Brewster's angle, which is defined by:

$$\tan(\theta_B) = n_2/n_1 \tag{9}$$

so, $\theta_B = \arctan(n_2/n1) = 0.90$ Rad.

ELECTROMAGNETISM 3

Charged sphere in magnetic field



A point charge $(-Q)$ is surrounded by a thin non-conducting sphere of radius R with evenly distributed charge Q so that the entire system is neutral. The sphere has total mass M and can freely rotate around the center without friction. Initially, there is uniform external magnetic field B along the $\hat{z}$ axis, and the sphere is static.

(a)[5pt] Calculate the Poynting vector $\mathbf{S}$ at every point in space and the total angular momentum carried by the electromagnetic fields.
Hint: Spherical coordinates and basis $(\hat{r}, \hat{\theta}, \hat{\varphi})$ are the most convenient choice.

(b)[5pt] The external magnetic field is gradually reduced to zero. Calculate the final angular velocity ω of the sphere. Neglect the magnetic field created by the sphere itself.
Hint: The moment of inertia of a thin sphere is $I = \frac{2}{3}MR^2$.

(c)[2pt] Compare the result of part (a) to part (b) and provide a qualitative explanation.

(d)[8pt] Calculate the magnetic field inside and outside of the charged sphere rotating with angular velocity $\omega\hat{z}$. One potential approach is

i) calculate the magnetic field $\mathbf{B}_1$ in the center of the rotating sphere;

ii) calculate the total magnetic dipole moment $\mathbf{m}$ of the rotating sphere;

iii) assuming that the magnetic field outside the sphere is that of a *point-like magnetic dipole* $\mathbf{m}$, calculate the magnetic field $\mathbf{B}_2(r, \theta, \varphi)$ for $r > R$;

iv) show that the magnetic field equal to $\mathbf{B}_2$ *just outside* the sphere and $\mathbf{B}_1$ *just inside* the sphere satisfies the Maxwell equations.

Alternative methods to calculate the magnetic field will also be accepted for full score.

Solution

(a) The electric field inside the sphere is radial, $\mathbf{E} = -\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$, and the Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{QB\sin\theta}{4\pi\epsilon_0\mu_0 r^2}\hat{\varphi}\,, \tag{1}$$

where in the last line we have used $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$. The density of mechanical momentum is $\pi = \mathbf{S}/c^2$, and the density of angular momentum (e.g., around the origin) is

$$\mathbf{m} = \mathbf{r}\times\pi = -\frac{QB\sin\theta}{4\pi r}\hat{\theta} \tag{2}$$

Averaging this vector over the volume of the sphere yields nonzero value only along the $\hat{z}$ axis, so we need to integrate only the $\hat{\theta}_z = -\sin\theta$ component of $\mathbf{m}$:

$$\begin{aligned} M_z &= \int_0^R dr\, r^2 \int_0^\pi d\theta\, \sin\theta \int_0^{2\pi} d\varphi\, \frac{QB\sin\theta}{4\pi r}\, \sin\theta \\ &= \frac{1}{2}QB\int_0^R dr\, r\int_0^\pi d\theta\, \sin^3\theta = \frac{1}{3}QBR^2\,. \end{aligned} \tag{3}$$

(b) As the field is reduced, there will be induced electric field moving the charges in the direction to compensate the change in the field, so that the total induced electric field around the $\hat{z}$ axis of the sphere will be

$$\oint_{\partial A} d\mathbf{l}\cdot\mathbf{E}_{\mathrm{ind}} = A\left(-\frac{\partial B}{\partial t}\right)\,, \tag{4}$$

where ∂A is the edge of a surface patch A. Taking A to be a cross-section of the sphere at angle θ, the integral above becomes $(\pi R^2\sin^2\theta)(-\partial B/\partial t)$, and the total torque acting on the ring-like segment of the sphere $[\theta;\theta+d\theta]\times[0;2\pi]$ is

$$d\tau = \sigma(Rd\theta)\,(R\sin\theta)\,(\pi R^2\sin^2\theta)\left(-\frac{\partial B}{\partial t}\right) \tag{5}$$

where $\sigma = \frac{Q}{4\pi R^2}$ is the surface charge density and $R\sin\theta$ is the "lever arm". Carrying out the integration over θ yields

$$\tau = \frac{Q}{4\pi R^2}\pi R^4\int_0^\pi d\theta\,\sin^3\theta\left(-\frac{\partial B}{\partial t}\right)\hat{z} = \frac{1}{3}QR^2\left(-\frac{\partial B}{\partial t}\right)\hat{z}\,. \tag{6}$$

Changing the field from $B\hat{z}$ to 0 yields the total change of the sphere's angular momentum to $\int d\tau = \frac{1}{3}QR^2(-\Delta\mathbf{B}) = \frac{1}{3}QBR^2\hat{z}$, so its final angular velocity is

$$\omega = \frac{\mathbf{L}}{I} = \frac{(1/3)QBR^2\mathbf{z}}{(2/3)MR^2} = \frac{QB}{2M}\mathbf{z} \tag{7}$$

(c) The angular momentum of the electromagnetic fields was transferred to the sphere. It is possible to demonstrate that by computing the full Maxwell's stress tensor that contains the shear stress (flow of angular momentum).

(d)

i) The magnetic field in the center of the sphere can be computed using the Biot-Savart law. It will be aligned with the angular velocity and the $\hat{z}$ axis. The ring-like spherical segment $[\theta;\theta+d\theta]\times[0;2\pi]$ carries current $dI = \sigma R d\theta\, v = \sigma R d\theta\, \omega R \sin\theta = (Q\omega/4\pi)\sin\theta d\theta$, and its contribution to the magnetic field in the center is

$$dB_{1z} = \frac{\mu_0}{4\pi R^2}(l dI)\,\sin\theta = \frac{\mu_0}{4\pi R^2}\frac{Q\omega}{4\pi}\sin\theta d\theta\,(2\pi R\sin\theta)\,\sin\theta = \frac{\mu_0 Q\omega}{8\pi R}\sin^3\theta\, d\theta \tag{8}$$

and the integration yields $B_{1z} = \frac{\mu_0 Q\omega}{6\pi R}$.

ii) The total magnetic dipole can be likewise found by integration over the sphere; the segment contribution is

$$dm_z = SdI = \pi R^2 \sin^2\theta\,\frac{1}{4\pi}Q\omega\sin\theta d\theta = \frac{1}{4}Q\omega R^2\sin^3\theta d\theta \tag{9}$$

and the integrated value is $m_z = \frac{1}{3}Q\omega R^2$.

iii) The magnetic dipole field is

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi R^3}\left[3(\mathbf{m}\cdot\hat{r})\hat{r} - \mathbf{m}\right] = \frac{\mu_0 Q\omega}{12\pi R}\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right] \tag{10}$$

iv) The difference between the magnetic dipole field $\mathbf{B}_2$ and the constant field $\mathbf{B}_1 = B_1\hat{z} = \frac{\mu_0 Q\omega}{6\pi R}(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$ is

$$\mathbf{B}_2 - \mathbf{B}_1 = \frac{\mu_0 Q\omega}{4\pi R}\sin\theta\,\hat{\theta} \tag{11}$$

which is tangential and exactly satisfies the Maxwell equation $(\mathbf{n}\cdot\nabla)\times\mathbf{B} = \mu_0\mathbf{K}$ with the surface current density $\mathbf{K} = \sigma\mathbf{v} = Q\omega\sin\theta\hat{\varphi}/(4\pi R)$, where $\mathbf{n} = \hat{r}$ is the outward normal vector to the sphere's surface.

QUANTUM MECHANICS 1

Hyperbolic cosine quantum well.

A particle of mass m moves in a 1D hyperbolic cosine quantum well with potential energy

$$V(x) = -\frac{\hbar^2}{ma^2}\frac{1}{\cosh^2(x/a)},$$

where a is the "size" of the well.

(a)[6pt] Show that $\psi_0(x) = C/\cosh(x/a)$ is a bound state solution, i.e., it solves the time-independent Schrödinger equation. This turns out to be the only bound state; all excited states have positive energies and are, therefore, the extended states. Find the exact bound state energy E_0 and normalize the wave-function $\psi_0(x)$.

(b)[4pt] Two identical spin-1/2 particles occupy the state $\psi_0(x)$. Neglect first the interaction between the particles. For the ground state of this system, write down the two-particle wave-function $\psi(x_1, x_2)\,\chi(1,2)$, including the spin part $\chi(1,2)$. What is the total energy of this state? Is it degenerate?

(c)[5pt] Assume now that the two particles interact via potential $U(x_1, x_2) = U_0 a\delta(x_1 - x_2)$. Evaluate the correction δE to the ground state energy due to this interaction using the first-order perturbation theory.

(d)[5pt] If the particles interact also through an anisotropic spin interaction

$$V_S = U_1\sigma_{z1}\sigma_{z2} + U_2[\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2}],$$

where $U_{1,2} > 0$, find the corresponding contribution E_S to the ground-state energy of the two-particle system.

The following integral might be useful in this problem:

$$\int \frac{dx}{\cosh^n(x)} = \frac{\sinh(x)}{(n-1)\cosh^{(n-1)}(x)} + \frac{n-2}{n-1}\int \frac{dx}{\cosh^{(n-2)}(x)}.$$

Solution

(a) The stationary Scrödinger equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \frac{\hbar^2}{ma^2}\frac{1}{\cosh^2(x/a)}\psi(x) = E\psi(x)\,.$$

For $\psi_0(x) = C/\cosh(x/a)$, we have

$$\frac{d\psi_0}{dx} = -\frac{C\sinh(x/a)}{a\cosh^2(x/a)}\,,$$

$$\frac{d^2\psi_0}{dx^2} = -\frac{C}{a^2\cosh(x/a)} + 2\frac{C\sinh^2(x/a)}{a^2\cosh^3(x/a)} = \frac{C}{a^2\cosh(x/a)} - \frac{2C}{a^2\cosh^3(x/a)}\,.$$

We see that the equation is satisfied, if

$$E_0 = -\frac{\hbar^2}{2ma^2}\,.$$

Using the third of the given integrals, we find directly the normalization constant

$$C = \frac{1}{\sqrt{2a}}\,.$$

(b) Since both particles occupy the same ground state, the coordinate part of the two-particle wave-function should simply be the product of two ψ_0s:

$$\psi(x_1, x_2) = \psi_0(x_1)\psi_0(x_2)\,. \tag{1}$$

The spin-1/2 particles are fermions, for which the total wave-function should be antisymmetric with respect to the interchange of the particle coordinates. The coordinate part (1) of the wave-function is symmetric with respect to the interchange of x_1, x_2. This means that the spin part should be antisymmetric:

$$\chi(1,2) = -\chi(2,1)\,.$$

As one knows from the basic addition of the quantum angular momenta, the antisymmetric combination of the two spin-1/2 is the "singlet" state which has zero total angular momentum

$$|\chi(1,2)\rangle = \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle)\,.$$

The singlet state is unique. This means that the ground state of the two-particle system is not degenerate. The energy E of this state is just the sum of the energies of the two single-particle states

$$E = 2E_0 = -\frac{\hbar^2}{ma^2}\,.$$

(c) Correction to energy due to perturbation U in the first order of the perturbation theory is given by the following standard expression

$$\delta E = \langle \psi | U | \psi \rangle = \int dx_1 dx_2 \psi^*(x_1, x_2) U(x_1, x_2) \psi(x_1, x_2) \,.$$

where we took into account that the perturbation potential does not act on spins. Using expressions for the perturbation potential and the two-particle wave-function, we get:

$$\delta E = U_0 a \int dx [\psi_0(x)]^4 = \frac{U_0}{4a} \int \frac{dx}{[\cosh(x/a)]^4} \,.$$

Making use of the integral given in the problem, we find

$$\delta E = \frac{1}{3} U_0 \,.$$

(d) One can see directly that the singlet spin state is an eigenstate of the anisotropic spin interaction. Indeed, for the z-component part of the interaction, one has

$$\sigma_{z1}\sigma_{z2}|\chi(1,2)\rangle = -|\chi(1,2)\rangle \,.$$

The same can be seen directly for the x, y-component of the interaction, by introducing the usual spin-flip operators

$$\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y) \,, \quad \text{i.e.} \quad \sigma_x = \sigma_+ + \sigma_- \,, \quad \sigma_y = -i(\sigma_+ - \sigma_-) \,.$$

which have the following properties

$$\sigma_+ | \downarrow\rangle = | \uparrow\rangle \,, \quad \sigma_- | \uparrow\rangle = | \downarrow\rangle \,.$$

In terms of these operators, the x, y-component of the interaction is:

$$\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2} = 2[\sigma_{+,1}\sigma_{-,2} + \sigma_{-,1}\sigma_{+,2}] \,.$$

From this, we see immediately that the x, y-component of the interaction acts on the singlet state similarly to the z-component:

$$(\sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2})|\chi(1,2)\rangle = -2|\chi(1,2)\rangle.$$

This means that the singlet state is an eigenstate of the spin interaction V_S with the eigenvalue E_S:

$$E_S = -(U_1 + 2U_2) \,.$$

An alternative solution is to construct a $4x4$ matrix of the 2-spin Hamiltonian,

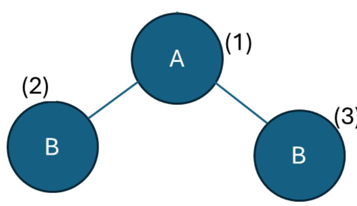
$$V_S = \begin{pmatrix} U_1 & & & \\ & -U_1 & 2U_2 & \\ & 2U_2 & -U_1 & \\ & & & U_1 \end{pmatrix} ,$$

which is easy to diagonalize to find the eigenvalues U_1 ($\times 2$ degenerate) and $(-U_1 \pm 2U_2)$.

QUANTUM MECHANICS 2

Toy Model of Molecule

In a highly simplified model of a molecule, an electron moves in the vicinity of three nuclei which include one "A" nucleus and two "B" nuclei, inter-linked as shown in the figure. In absence of any coupling between electron states bound to different nuclei, the electron has energy $E_1^{(0)}$ when it is on the "A" nucleus and energy $E_2^{(0)}$ when it is on a "B" nucleus, such that $E_2^{(0)} - E_1^{(0)} = \Delta > 0$. The state of the electron can be defined by its location, labeled as (1)-(3) in the figure: $|1\rangle = (100)$, $|2\rangle = (010)$, $|3\rangle = (001)$.



(a)[2pt] Under the approximation that the electron cannot move from one nuclei to another, write down the Hamiltonian in the basis of state $|1\rangle$, $|2\rangle$, and $|3\rangle$.

(b)[2pt] Now we will allow the electron to move between an A atom and a B atom with a coupling energy of a,but not between the B atoms. Assume $a \ll \Delta$. Write down the perturbation Hamiltonian associated with A-B coupling.

(c)[5pt] Calculate perturbation to energy $E_1^{(0)}$ to the second order and calculate the 1st order perturbation to the energy $E_2^{(0)}$.

(d)[6pt] Now calculate the precise (without using perturbation method) eigenstates of the system and their energies. Compare your result with the perturbation calculation under the limit $a \ll \Delta$.

(e)[5pt] Assuming an electron at the vicinity of nuclei A at $t = 0$, what is the probability of finding the electron at the vicinity of a nuclei B at time t?

Solution

(a) Without inter-nuclei coupling, the Hamiltonian is simply:

$$H_0 = \begin{pmatrix} E_1^{(0)} & 0 & 0 \\ 0 & E_2^{(0)} & 0 \\ 0 & 0 & E_2^{(0)} \end{pmatrix} \tag{1}$$

(b) The perturbation Hamiltonian describes the couplings between the states:

$$W_0 = \begin{pmatrix} 0 & a & a \\ a & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \tag{2}$$

(c) The energy state of $E_1^{(0)}$ is non-degenerate, and the energy correction up to the second-order perturbation is:

$$E_1^{(2)} = E_1^{(0)} + \langle 1|W|1\rangle + \sum_{i=2}^{3} \frac{|\langle i|W|1\rangle|^2}{E_1^{(0)} - E_2^{(0)}} = E_1^{(0)} - \frac{2a^2}{\Delta} \tag{3}$$

The states $|2\rangle$, $|3\rangle$ are degenerate. The matrix elements of W between the two $E_2^{(0)}$ states are zero. The secular equation is:

$$det \begin{pmatrix} 0-\epsilon & 0 \\ 0 & 0-\epsilon \end{pmatrix} = 0 \tag{4}$$

Hence $\epsilon = 0$ and the first-order correction to the $E_2^{(0)}$ is zero.

(d) The full Hamiltonian is:

$$H = \begin{pmatrix} E_1^{(0)} & a & a \\ a & E_2^{(0)} & 0 \\ a & 0 & E_2^{(0)} \end{pmatrix} \tag{5}$$

To find the eigen-energies of the system we solve the equation $\det(H - EI) = 0$. We have

$$(E_1^{(0)} - E)(E_2^{(0)} - E)^2 - 2a^2(E_2^{(0)} - E) = 0 \tag{6}$$

hence

$$(E_2^{(0)} - E)[(E_1^{(0)} - E)(E_2^{(0)} - E) - 2a^2] = 0 \tag{7}$$

Solve for E, we have:

$$E_{1,3} = \frac{1}{2}\left[(E_1^{(0)} + E_2^{(0)}) \pm \sqrt{\Delta^2 + 8a^2}\right] \tag{8}$$

$$E_2 = E_2^{(0)} \tag{9}$$

Under the limit $a \ll \Delta$,

$$\begin{aligned} E_{1,3} &= \frac{1}{2}\left[(E_1^{(0)} + E_2^{(0)}) \pm \Delta\sqrt{1 + \frac{8a^2}{\Delta^2}}\right] \\ &\simeq \frac{1}{2}\left[(E_1^{(0)} + E_2^{(0)}) \pm \Delta(1 + \frac{4a^2}{\Delta^2})\right] \end{aligned}$$

Hence we get:

$$E_1 \simeq E_2^{(0)} + \frac{2a^2}{\Delta} \tag{10}$$

$$E_3 \simeq E_1^{(0)} - \frac{2a^2}{\Delta} \tag{11}$$

Note that the result for E_1 is the same as that in the perturbation calculation under the limit $a \ll \Delta$.

The eigenvector $\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$ follows:

$$(E_1^{(0)} - E)\phi_1 + a(\phi_2 + \phi_3) = 0 \tag{12}$$

$$a\phi_1 + (E_2^{(0)} - E)\phi_2 = 0 \tag{13}$$

$$a\phi_1 + (E_2^{(0)} - E)\phi_3 = 0 \tag{14}$$

With $E = E_1, E_3$ this gives

$$\begin{aligned} \psi_{1,3} &= \frac{1}{\sqrt{1 + \frac{2a^2}{(E_2^{(0)} - E_{1,3})^2}}} \begin{pmatrix} 1 \\ \frac{-a}{E_2^{(0)} - E_{1,3}} \\ \frac{-a}{E_2^{(0)} - E_{1,3}} \end{pmatrix} \\ &= \frac{1}{\sqrt{1 \mp \frac{8a^2}{(\Delta - \sqrt{\Delta^2 + 8a^2})^2}}} \begin{pmatrix} 1 \\ \frac{-2a}{\Delta \mp \sqrt{\Delta^2 + 8a^2}} \\ \frac{-2a}{\Delta \mp \sqrt{\Delta^2 + 8a^2}} \end{pmatrix} \end{aligned}$$

And with $E = E_2$:

$$\psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{15}$$

(e) The full wave-function is a superposition of the eigenstates:

$$\Phi = \sum_{i=1}^{3} c_i \psi_i e^{-\frac{iE_i t}{\hbar}} \tag{16}$$

at $t = 0$ the electron is at the vicinity of A nuclei, so

$$\Phi(0) = \sum_{i=1}^{3} c_i \psi_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{17}$$

For simplicity we define the normalization factors as $\frac{1}{\sqrt{1 \mp \frac{8a^2}{(\Delta - \sqrt{\Delta^2 + 8a^2})^2}}} = \alpha_{1,3}$r, and $\frac{-2a}{\Delta \mp \sqrt{\Delta^2 + 8a^2}} = \gamma_{1,3}$ so

$$\psi_{1,3} = \alpha_{1,3} \begin{pmatrix} 1 \\ \gamma_{1,3} \\ \gamma_{1,3} \end{pmatrix} \tag{18}$$

The equations for c_i are:

$$c_1 \alpha_1 + c_3 \alpha_3 = 1 \tag{19}$$

$$c_1 \alpha_1 \gamma_1 + \frac{c_2}{\sqrt{2}} + c_3 \alpha_3 \gamma_3 = 0 \tag{20}$$

$$c_1 \alpha_1 \gamma_1 - \frac{c_2}{\sqrt{2}} + c_3 \alpha_3 \gamma_3 = 0 \tag{21}$$

We get $c_1 = \frac{\gamma_3}{\alpha_1(\gamma_3 - \gamma_1)}$, $c_2 = 0$ and $c_3 = \frac{\gamma_1}{\alpha_3(\gamma_1 - \gamma_3)}$.

At time t, the wave-function is

$$\Phi(t) = \begin{pmatrix} c_1\alpha_1 e^{-\frac{iE_1t}{\hbar}} + c_3\alpha_3 e^{-\frac{iE_3t}{\hbar}} \\ c_1\alpha_1\gamma_1 e^{-\frac{iE_1t}{\hbar}} + c_3\alpha_3\gamma_3 e^{-\frac{iE_3t}{\hbar}} \\ c_1\alpha_1\gamma_1 e^{-\frac{iE_1t}{\hbar}} + c_3\alpha_3\gamma_3 e^{-\frac{iE_3t}{\hbar}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\gamma_3}{\gamma_3-\gamma_1} e^{-\frac{iE_1t}{\hbar}} + \frac{\gamma_1}{\gamma_1-\gamma_3} e^{-\frac{iE_3t}{\hbar}} \\ \frac{\gamma_1\gamma_3}{\gamma_3-\gamma_1} e^{-\frac{iE_1t}{\hbar}} + \frac{\gamma_1\gamma_3}{\gamma_1-\gamma_3} e^{-\frac{iE_3t}{\hbar}} \\ \frac{\gamma_1\gamma_3}{\gamma_3-\gamma_1} e^{-\frac{iE_1t}{\hbar}} + \frac{\gamma_1\gamma_3}{\gamma_1-\gamma_3} e^{-\frac{iE_3t}{\hbar}} \end{pmatrix}$$

$$= \frac{\gamma_1\gamma_3}{\gamma_3-\gamma_1} \begin{pmatrix} \frac{1}{\gamma_1} e^{-\frac{iE_1t}{\hbar}} - \frac{1}{\gamma_3} e^{-\frac{iE_3t}{\hbar}} \\ e^{-\frac{iE_1t}{\hbar}} - e^{-\frac{iE_3t}{\hbar}} \\ e^{-\frac{iE_1t}{\hbar}} - e^{-\frac{iE_3t}{\hbar}} \end{pmatrix}$$

The probability of finding the electron at the vicinity of a B nuclei is:

$$P = 2(\frac{\gamma_1\gamma_3}{\gamma_3-\gamma_1})^2 |e^{-\frac{iE_1t}{\hbar}} - e^{-\frac{iE_3t}{\hbar}}|^2 \tag{22}$$

$$P = 4(\frac{\gamma_1\gamma_3}{\gamma_3-\gamma_1})^2 (1 - \cos\frac{(E_1-E_3)t}{\hbar}) \tag{23}$$

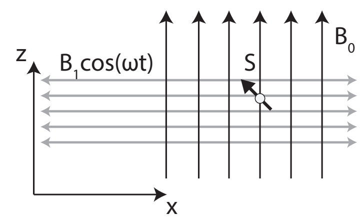
Put in the values of $\gamma_{1,3}$, this is

$$P = \frac{4a^2}{\Delta^2 + 8a^2} (1 - \cos\frac{(E_1-E_3)t}{\hbar}) \tag{24}$$

QUANTUM MECHANICS 3

Paramagnetic resonance

Consider a static electron in a constant magnetic field of magnitude B_0 along the z axis. In addition, there is a time-dependent magnetic field along the x axis with magnitude $B_1 \cos \omega t$, as shown in the figure, where ω is the typical frequency of the field. Let's assume that $B_0 > B_1$



(a)[3pt] Calculate the Larmor frequency ω_L of the electron in the constant magnetic field B_0. *(The Larmor frequency is the frequency of electron spin precession in a constant magnetic field)*

(b)[6pt] Write the time-dependent Schrödinger equation for the two spin projections of the electron.

(c)[5pt] Find the general solution for the time-dependent Schrödinger equation for the two spin projections *using the rotating wave approximation*, in which only terms with frequency $(\omega - \omega_L)$ contribute to the spin dynamics (in other words, the fast oscillations in spin dynamics are neglected). Can you comment on the validity and accuracy of the approximation?

(d)[3pt] Assuming that the electron at t=0 is the $|S_z = \frac{1}{2}\rangle$ spin state, what is the probability of finding the electron in the $|S_z = -\frac{1}{2}\rangle$ spin state at time T?

(e)[3pt] Can you sketch the probability of finding the electron in the $|S_z = -\frac{1}{2}\rangle$ state as a function of ω for some fixed time T? What does it look like? Does it resemble some well-known function?

As a reminder, here are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} ; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Solution

(a) The Larmor frequency is the precession frequency of the spin of a particle in the presence of a constant magnetic field. In the case of the electron, the Larmor frequency is $\omega_L = \frac{g_e \mu_B B_0}{\hbar}$, where μ_B is the Bohr magneton and g_e is the electron gyromagnetic ratio. This result is easily obtained from dimensional analysis, keeping in mind that the Hamiltonian of the system is $\hat{H} = g_e \mu_B \mathbf{B} \cdot \hat{\mathbf{S}}/\hbar$.

(b) Let's denote the spin-up projection of the electron spin by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1}$$

and the spin-down as

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2}$$

In this case, assuming that the wave function of the electron is given by

$$|\Psi(t)\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}, \tag{3}$$

one finds that the time-dependent Shchröringer equation $\imath \frac{\partial |\Psi(t)\rangle}{\partial t} = \hat{H}|\Psi(t)\rangle$, yields

$$\frac{d\alpha(t)}{dt} = -\imath \frac{\omega_L}{2}\alpha(t) - \imath \frac{\omega_L B_1}{2B_0}\cos(\omega t)\beta(t) \tag{4}$$

$$\frac{d\beta(t)}{dt} = \imath \frac{\omega_L}{2}\beta(t) - \imath \frac{\omega_L B_1}{2B_0}\cos(\omega t)\alpha(t) \tag{5}$$

$$\tag{6}$$

(c) To find the general solution of the time-dependent Shchröringer equation, we introduce the following changes of variables: $\alpha(t) = a(t)e^{-\imath\omega_L t/2}$ and $\beta(t) = b(t)e^{\imath\omega_L t/2}$, yielding

$$\frac{da(t)}{dt} = -\imath \frac{\omega_L B_1}{2B_0}\cos(\omega t)b(t) \tag{7}$$

$$\frac{db(t)}{dt} = -\imath \frac{\omega_L B_1}{2B_0}\cos(\omega t)a(t) \tag{8}$$

$$\tag{9}$$

At this point, we use the rotating wave approximation; only terms $\omega_L - \omega$ are kept, and hence, one finds

$$\frac{da(t)}{dt} = -\imath\frac{\omega_L B_1}{4B_0}e^{\imath(\omega_L-\omega)t}b(t) \tag{10}$$

$$\frac{db(t)}{dt} = -\imath\frac{\omega_L B_1}{4B_0}e^{\imath(\omega-\omega_L)t}a(t) \tag{11}$$

$$. \tag{12}$$

Next, taking the second derivative with respect of time of $a(t)$ and using the differential equation for $\frac{db(t)}{dt}$, we find

$$\frac{d^2a(t)}{dt^2} - \imath(\omega_L-\omega)\frac{da(t)}{dt} + \left(\frac{\omega_L B_1}{4B_0}\right)^2 a(t) = 0, \tag{13}$$

and the general solution is given by

$$a(t) = a_+e^{\imath\lambda_+ t} + a_-e^{\imath\lambda_- t}, \tag{14}$$

where

$$\lambda_\pm = \frac{1}{2}\left((\omega_L-\omega) \pm \sqrt{(\omega_L-\omega)^2 + 4\left(\frac{\omega_L B_1}{4B_0}\right)^2}\right). \tag{15}$$

Similarly,

$$b(t) = \imath\frac{4B_0}{B_1\omega_L}e^{\imath(\omega-\omega_L)t}\left(\imath\lambda_+a_+e^{\imath\lambda_+ t} + \imath\lambda_-a_-e^{\imath\lambda_- t}\right). \tag{16}$$

(d) The initial conditions of the problem implies that $a_+ + a_- = 1$ and $\lambda_+a_+ + a_-\lambda_- = 0$. The solution is $a_+ = \frac{-\lambda_-}{\lambda_+-\lambda_-}$ and $a_- = \frac{\lambda_+}{\lambda_+-\lambda_-}$. With this in mind, we can use Eq. (16) to get the probability of finding the electron in the spin-down projection at time T as

$$|b(T)|^2 = \frac{4\left(\frac{\omega_L B_1}{4B_0}\right)^2}{(\omega_L-\omega)^2 + 4\left(\frac{\omega_L B_1}{4B_0}\right)^2}\sin^2\left(T/2\sqrt{(\omega_L-\omega)^2 + 4\left(\frac{\omega_L B_1}{4B_0}\right)^2}\right). \tag{17}$$

The probability at a given time T follows a Lorentzian profile.

(e) As explained before, the probability of excitation follows a Lorentzian profile as a function of the driven frequency ω, and hence it looks like

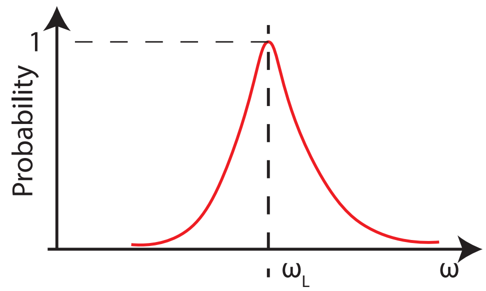
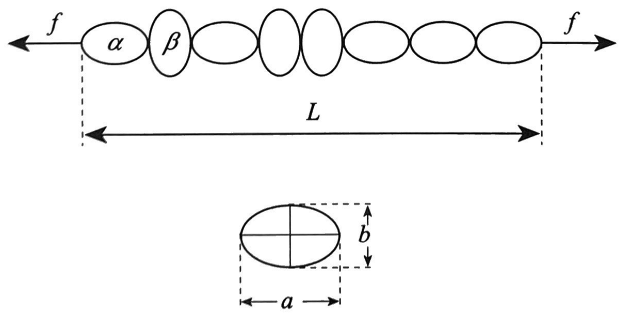


FIG. 2. Probability of finding the electron in the spin-down projection as a function of the driven frequency

STATISTICAL MECHANICS 1

Molecular chain



Consider a one-dimensional chain of molecules consisting of N molecules which exist in two configurations α and β with corresponding energies ϵ_α and ϵ_β and lengths a and b. The chain is in thermal equilibrium at temperature τ and subject to a tensile force f (see the figure).

(a)[5pt] Find the partition function.

(b)[5pt] Find the average length $\langle L \rangle$ of the chain as a function of f and temperature.

(c)[5pt] Assume that $\epsilon_\alpha > \epsilon_\beta$ and $a > b$. Estimate the average length $\langle L \rangle$ in the absence of the tensile force as a function of temperature. What are the high and low temperature limits? What do these correspond to physically? Sketch a plot of $\langle L \rangle$ versus τ.

(d)[5pt] Calculate the linear response function (Hooke's constant) of the chain

$$\chi = \left(\frac{\partial \langle L \rangle}{\partial f} \right)_{f=0}$$

and show that it is greater than zero. Why should this be the case?

Solution

(a) Consider one link of the chain in its two configurations α and β. The energy of the link in each configuration is $E_\alpha = \epsilon_\alpha - fa$ and $E_\beta = \epsilon_\beta - fb$. The partition function for the whole chain is given by

$$Z = \left(\sum_{\alpha,\beta} e^{-E_{\alpha,\beta}/\tau} \right)^N = \left(e^{(fa-\epsilon_\alpha)/\tau} + e^{(fb-\epsilon_\beta)/\tau} \right)^N \tag{1}$$

(b) The average length of the chain may be found from the partition function:

$$\langle L \rangle = \tau \left(\frac{\partial \ln Z}{\partial f} \right)_{\tau,N} = \frac{N \left[a e^{(fa-\epsilon_\alpha)/\tau} + b e^{(fb-\epsilon_\beta)/\tau} \right]}{e^{(fa-\epsilon_\alpha)/\tau} + e^{(fb-\epsilon_\beta)/\tau}} \tag{2}$$

(c) If $f = 0$, Eq. (2) becomes

$$\langle L \rangle = N \frac{a e^{-\epsilon_\alpha/\tau} + b e^{-\epsilon_\beta/\tau}}{e^{-\epsilon_\alpha/\tau} + e^{-\epsilon_\beta/\tau}} = N \frac{a + b e^{(\epsilon_\alpha - \epsilon_\beta)/\tau}}{1 + e^{(\epsilon_\alpha - \epsilon_\beta)\tau}} \tag{3}$$

If $\epsilon_{\alpha,\beta} \ll \tau$ (high temperature):

$$\langle L \rangle \simeq N \frac{a+b}{2} \tag{4}$$

which indicates that on average half of the links are in configuration α and half are in β. If $\epsilon_{\alpha,\beta} \gg \tau$ (low temperature):

$$\langle L \rangle \simeq N[a e^{-(\epsilon_\alpha - \epsilon_\beta)/\tau} + b] \tag{5}$$

So almost all links in the chain are in β. The changeover temperature is $\epsilon_\alpha - \epsilon_\beta$.

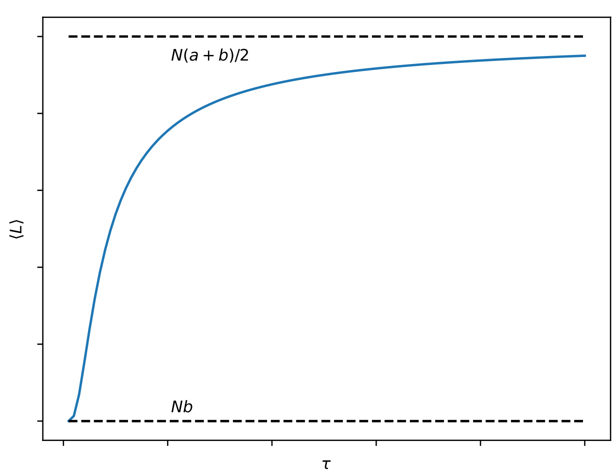


FIG. 3. Plot of average length versus temperature.

(d) We can rewrite Eq. (2) using $\delta = (\epsilon_\alpha - \epsilon_\beta)/\tau$

$$\langle L \rangle = N \frac{a e^{fa/\tau} + b e^{fb/\tau + \delta}}{e^{fa/\tau} + e^{fb/\tau+\delta}} = N \frac{a + b e^{f(b-a)/\tau+\delta}}{1 + e^{f(b-a)\tau+\delta}}. \tag{6}$$

At small f this becomes

$$\begin{aligned}
\langle L \rangle &\simeq N \frac{a + be^\delta + e^\delta b f(b-a)/\tau}{1 + e^\delta + e^\delta f(b-a)/\tau} \\
&\simeq N \left[\frac{a + be^\delta}{1+e^\delta} + \frac{e^\delta b f(b-a)/\tau}{1+e^\delta} - \frac{e^\delta (a+be^\delta) f(b-a)/\tau}{(1+e^\delta)^2} \right] \\
&= N \frac{a+be^\delta}{1+e^\delta} + N \frac{e^\delta f(b-a)}{\tau(1+e^\delta)} \left[b - \frac{a+be^\delta}{1+e^\delta} \right] \\
&= N \frac{a+be^\delta}{1+e^\delta} + N \frac{e^\delta f(b-a)^2}{\tau(1+e^\delta)^2}.
\end{aligned} \tag{7}$$

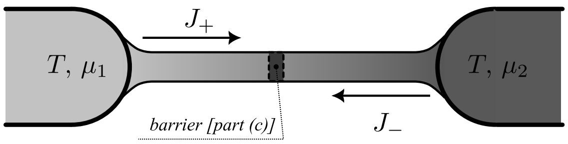
Therefore,

$$\left. \frac{\partial \langle L \rangle}{\partial f} \right|_{f=0} = \frac{N e^\delta}{\tau} \left(\frac{b-a}{1+e^\delta} \right)^2 > 0 \tag{8}$$

as it should be since it corresponds to the thermodynamic inequality for a system in equilibrium: $-(\partial V/\partial P) > 0$.

STATISTICAL MECHANICS 2

Conductance quantization and counting statistics



In this problem, we discuss electron transport properties of a narrow ballistic channel between two *large* equilibrium electrodes each in equilibrium with different chemical potentials $\mu_{1,2}$ and small temperature $T \ll \mu_{1,2}$. As a result, the voltage across the channel is equal to $V = (\mu_1 - \mu_2)/e$. The channel can be modelled as a one-dimensional (1D) gas of non-interacting electrons with the two different Fermi energies μ_1 and μ_2 on the two sides of the 1D Fermi surface. Neglect electron spin.

(a)[4pt] Assuming a general dispersion relation $\epsilon(k)$ for the electrons, calculate the density of single-particle states $\nu(\epsilon) = dn/d\epsilon$ per unit length in the channel (e.g., with the periodic boundary conditions on some normalization length L). Do this separately at the two end points of the 1D Fermi surface, which correspond to electrons moving in opposite directions.

(b)[5pt] Assuming steady equilibrium between the channel and the electrodes, justify which side of the Fermi surface (left- or right-moving) of electrons in the channel must be at chemical potentials μ_1 and μ_2. Write down the expressions for the electric currents $J_\pm$ carried by electrons in each direction. Calculate the net current $J = J_+ - J_-$ through the channel for small bias voltage $eV \ll \mu_{1,2}$, show that it satisfies Ohm's law $J = GV$, and find the conductance G of the channel. *Hint: Electric current carried by electrons with energy ϵ is equal to $en(\epsilon)u(\epsilon)$, where $n(\epsilon)$ is the 1D concentration of electrons with energy ϵ, and $u(\epsilon) = (1/\hbar)d\epsilon/dk$ is the velocity of electrons with this energy.*

(c)[3pt] How the result for G in part (b) changes if the channel contains a potential barrier for electrons with probabilities of transmission D and reflection R?

(d)[4pt] At $T = 0$, one can find the probability distribution ("counting statistics") for the charge transferred through the channel using the following simple model. Non-interacting electrons in the energy range relevant for transport are incident on the tunnel barrier with some frequency f. Each electron scatters independently with the probabilities D and R to be transmitted through or reflected from the barrier, respectively. In this model, calculate the probability p_n that n electrons are transferred through the channel during a long time interval t, $N = tf \gg 1$. Calculate also the average current J and compare it to the results of parts (b) and (c) to find the frequency f.

(e)[4pt] Use the probability distribution p_n to calculate the stochastic noise in the current J. The noise is characterized by the zero-frequency spectral density $S_J(0)$ that is defined as $S_J(0) = e^2\sigma_n^2/t$, where σ_n is the standard deviation of n: $\sigma_n^2 = \langle n^2\rangle - \langle n\rangle^2$.

Solution

(a) With periodic boundary conditions imposed on some normalization length L, the single-particle momentum states are spaced by the wave-number interval $2\pi/L$. Therefore, the number of states dN in an interval dk of the wave-number is $dN = Ldk/(2\pi)$, and $dN = dk/(2\pi)$ per unit length. The wave-number interval dk correspond to the energy interval $d\epsilon = [d\epsilon(k)/dk]dk$. Hence, the energy density of the single-particle states per unit length is

$$\nu(\epsilon) = dN/d\epsilon = (dN/dk)(dk/d\epsilon) = 1/[2\pi d\epsilon(k)/dk]$$

at both of the two end points of the 1D Fermi surface (i.e., for both the forward- and backward-moving electrons).

(b) The equilibrium occupation probabilities of the single-particle states are given by the Fermi distribution functions $f(\epsilon, \mu_j)$. The 1D concentration of electrons with certain energy is then $n(\epsilon) = \nu(\epsilon) f(\epsilon, \mu_j)$, and the total current J_+ carried by the forward-moving electrons is

$$J_+ = e \int d\epsilon \nu(\epsilon) f(\epsilon, \mu_1) u(\epsilon) = \frac{e}{2\pi\hbar} \int d\epsilon f(\epsilon, \mu_1) \,.$$

The last equality follows since the factors $d\epsilon(k)/dk$ in the density of states and in the velocity cancel out. The current J_- carried by the backward-moving electrons is given by the same expression with μ_1 replaced by μ_2. The net current J is then

$$J = \int \frac{e}{h} \int d\epsilon [f(\epsilon, \mu_1) - f(\epsilon, \mu_2)] \,.$$

and for $eV, T \ll \mu_{1,2}$, this integral reduces simply to $(\mu_1 - \mu_2)$. We get, finally:

$$J = GV, \quad G = \frac{e^2}{h} \,.$$

We see that the conductance of the 1D ballistic channel is determined by the fundamental constants only.

(c) In the presence of the tunnel barrier with transmission probability D, only the fraction D of the incident electrons is transmitted on average through the channel. This means that the current is reduced by the factor D in comparison to the situation without the barrier, i.e.

$$G = \frac{e^2 D}{h} \,.$$

(d) The time interval t encloses $N = tf$ independent scattering events. In one scattering, an electron is transferred through the channel with probability D, and reflected from it with probability R, $D + R = 1$. The fact that different scattering events are independent immediately means that the probability p_n to have n electrons transferred in N attempts is given by the binomial distribution:

$$p_n = \frac{N!}{n!(N-n)!} D^n R^{N-n}.$$

The average n then is:

$$\langle n \rangle = \sum_{n=0}^{N} n p_n = \sum_{n=1}^{N} \frac{N!}{(n-1)!(N-n)!} D^n R^{N-n} = DN \sum_{k=0}^{N-1} \frac{(N-1)!}{k!(N-1-k)!} D^k R^{N-1-k} = DN.$$

This expression correspond precisely to the Ohm' law found in part (b) with

$$J = \frac{e\langle n \rangle}{t} = eDf\,.$$

Comparing this equation to the expression for the current found in parts (b) and (c), we see that the attempt frequency f is determined by the voltage across the channel:

$$f = eV/h\,.$$

(e) The standard deviation of n can be found by calculating first the average of the product $n(n-1)$, in precisely the same way as we just calculated the average n:

$$\langle n(n-1) \rangle = \sum_{n=0}^{N} n(n-1) p_n = \sum_{n=2}^{N} \frac{N!}{(n-2)!(N-n)!} D^n R^{N-n} = D^2 N(N-1).$$

From this

$$\sigma_n^2 = \langle n^2 \rangle - \langle n \rangle^2 = DN - D^2 N = NDR\,,$$

and finally

$$S_J(0) = \frac{e^3 V}{h} DR\,.$$

At small electron transparency, $D \ll 1$ this result reduces to classical "shot" noise $S_J(0) = eJ$ related to electron discreteness of the current flow. With increasing D, the shot noise is suppressed by quantum nature of electron propagation.

STATISTICAL MECHANICS 3

Quantum statistics

A potential well at temperature T contains exactly two non-interacting but identical fermions. The single-particle energy levels in the well are linearly spaced

$$\epsilon_k = k\Delta \quad \text{with} \quad k = 0, 1, \ldots \infty. \tag{1}$$

Each single-particle energy level consists of g quantum states. The g-fold degeneracy of each energy level comes from spin and other internal quantum numbers. Assume that $g \gg 1$. **(a)**[5pt] Calculate the free energy F of the system when $g \gg 1$, including the first correction

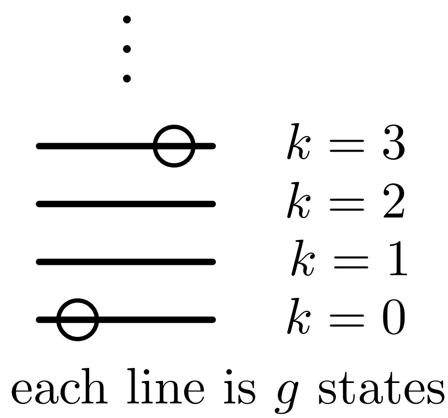


FIG. 4. Single-particle energy levels of a potential well. Each level (shown by the lines) consists of $g \gg 1$ internal states (not shown). The two circles indicate a configuration where one particle is one of the g-internal states with $k = 0$ and one particle is one of the g-internal states with $k = 3$.

due to quantum statistics. **(b)**[6pt] How would the result in part (a) change for identical bosons. Compute the difference in free energy, $\Delta F = F_+ - F_-$, where F_+ and F_- are the free energies of the bosons and fermions, respectively. Interpret the difference when $k_B T \ll \Delta$ by enumerating the possible states. Consider the same potential well with the same ϵ_k and degeneracies. Now, however, the well is in contact with a reservoir at temperature T and chemical potential μ, such that the average number of particles in the well is maintained at $\langle n \rangle = 2$. The number of particles in the well fluctuates around this mean value. **(c)**[6pt] Determine the fugacity $\lambda_\pm \equiv \exp(\beta\mu_\pm)$ for bosons $(+)$ and fermions $(-)$, including the leading correction due to quantum statistics when $g \gg 1$.

Hint: First find the fugacity at leading order, λ_0. Then find the first correction $\lambda = \lambda_0 + \delta\lambda$ iteratively. **(d)**[3pt] Compute the variance $\sigma_\pm^2$ in the number of particles in the well for bosons $(+)$ and fermions $(-)$, including the leading correction due to quantum statistics:

$$\sigma_\pm^2(T) \equiv \left\langle (\delta n)^2 \right\rangle . \tag{2}$$

Solution

(a) The single particle states are labeled by two integers $s_1 = (k_1, m_1)$ and $m_1 = 1 \dots g$. The s_1 and s_2 can't be equal for fermions.

$$
\begin{aligned}
Z =& \sum_{s_1, s_2 > s_1} e^{-\beta(k_1+k_2)\Delta} & (3)\\
=& \frac{1}{2}\sum_{k_1,k_2} g^2 e^{-\beta(k_1+k_2)\Delta} - \frac{1}{2}\sum_{k_1=k_2} g e^{-\beta 2k_1\Delta} & (4)\\
=& \frac{1}{2}(gz_1)^2 - \frac{1}{2} g z_2 & (5)
\end{aligned}
$$

Here the partition function of the harmonic oscillator is

$$
z_1 \equiv z(\Delta) = \frac{1}{1-e^{-\beta\Delta}} \qquad z_2 \equiv z(2\Delta) = \frac{1}{1-e^{-2\beta\Delta}} \tag{6}
$$

Then

$$
\begin{aligned}
F = -T\ln Z =& -T\ln\left[\frac{1}{2}(gz(\Delta))^2 - \frac{1}{2}gz(2\Delta)\right] & (7)\\
=& -2T\ln(gz_1) - T\ln(2) - T\ln(1 - \frac{z_2}{gz_1^2}) & (8)\\
\simeq& -2T\ln(gz_1) - T\ln(2) + \frac{Tz_2}{gz_1^2} & (9)
\end{aligned}
$$

(b) The Boson case has

$$
Z = \frac{1}{2}\sum_{k_1,k_2} g^2 e^{-\beta(k_1+k_2)\Delta} + \frac{1}{2}\sum_{k_1=k_2} g e^{-\beta 2k_1\Delta} \tag{10}
$$

(11)

and

$$
F_+ = -T\ln Z \simeq -2T\ln(gz_1) - T\ln(2) - \frac{Tz_2}{gz_1^2} \tag{12}
$$

Then

$$
\Delta F = -2T\frac{z_2}{gz_1^2} \tag{13}
$$

In the low temperature limit the z_2 and z_1 are one. The mean energy is zero (since both atoms are in $k=0$) and $F \simeq -TS = -2T/g$. The entropy is simply counting. All atoms are in the lowest energy level. There are $g(g+1)/2$ ways to put two bosons into g open states and $g(g-1)/2$ ways to put two fermions into g open states. Thus we find

$$
\Delta S = \ln(g(g+1)/2) - \ln(g(g-1)/2) \simeq \frac{2}{g}\,. \tag{14}
$$

(c) We will use the grand partition function formalism. The thing to recognize here is that because the occupancy $\bar{n}/g \ll 1$ is low, classical statistics are a good approximation. To set notation, the partition function is written

$$Z_G = \sum_s e^{-\beta E_s + \alpha N_s} \,, \tag{15}$$

where $\alpha = \mu/T$.

The mean number of particles follows from the Bose-Einstein or Fermi-Dirac distribution function:

$$\bar{n} = g \sum_k \frac{1}{e^{\beta\epsilon_k - \alpha} \mp 1} = g \sum_k \frac{\lambda e^{-\beta\epsilon_k}}{1 \mp \lambda e^{-\beta\epsilon_k}} \tag{16}$$

where we defined the fugacity $\lambda \equiv e^{\alpha}$. This could be written in a somewhat more familiar way:

$$\bar{n} = \sum_{states} \frac{1}{e^{\beta(\epsilon_k - \mu)} \mp 1} \tag{17}$$

Then since g is large here, the number of particles per quantum state is small, and we have that the fugacity is very small, $\lambda \equiv e^{\alpha} \ll 1$. This approximation is familiar from a classical ideal gas. We can expand Eq. 16 in λ and perform the sum

$$\bar{n} \simeq g \sum_k (\lambda e^{-\beta\epsilon_k} \pm \lambda^2 e^{-2\beta\epsilon_k}) = g\lambda z_1 + g\lambda^2 z_2 \,. \tag{18}$$

Thus the full equation which determines μ (or $\lambda \equiv e^{\beta\mu}$) is

$$\frac{\bar{n}}{g z_1} = \lambda \pm \lambda^2 \frac{z_2}{z_1} \,. \tag{19}$$

The first term on the right is small $O(e^{\alpha})$, while the second term is small squared $O(e^{2\alpha})$. Solving at lowest order we have

$$\lambda_0 = \frac{\bar{n}}{g z_1} \tag{20}$$

and the full equation reads

$$\lambda_0 = \lambda_\pm \pm \lambda_\pm^2 \frac{z_2}{z_1} \,. \tag{21}$$

Using the zero-th order solution we have approximately

$$\lambda_0 \simeq \lambda_\pm \pm \lambda_0^2 \frac{z_2}{z_1} \,, \tag{22}$$

and thus

$$\lambda_\pm \simeq \lambda_0 \mp \lambda_0^2 \frac{z_2}{z_1} \,. \tag{23}$$

(d) Then we can find the fluctuations

$$\langle \delta n^2 \rangle = \frac{\partial \bar{n}}{\partial \alpha} = g \sum_k e^{-\beta\epsilon_k + \alpha} \pm 2e^{2\alpha} e^{-2\beta\epsilon_k} = g z_1 \lambda \pm 2g\lambda^2 z_2 \tag{24}$$

$$= g z_1 \lambda_0 \pm g \lambda_0^2 z_2 \tag{25}$$

$$= g \left(\frac{\bar{n}}{g} \right) \left(1 \pm \frac{\bar{n}}{g} \frac{z_2}{z_1^2} \right) \tag{26}$$

Discussion: In the limit that $T \ll \Delta$ and we get the familiar result

$$\langle \delta n^2 \rangle = g \left[\frac{\bar{n}}{g} \left(1 \pm \frac{\bar{n}}{g} \right) \right] . \tag{27}$$

for the fluctuations of a Bose and Fermi gas. The difference between bosons and fermions is approximately

$$\Delta \sigma^2 = 2\bar{n} \left(\frac{\bar{n}}{g} \frac{z_2}{z_1^2} \right) \tag{28}$$